Advanced Computer Graphics
Generalized Barycentric Coordinates

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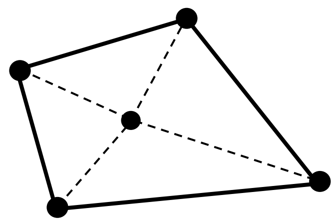
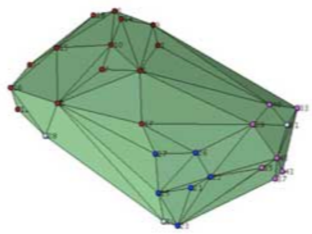
Generalized Barycentric Coordinates

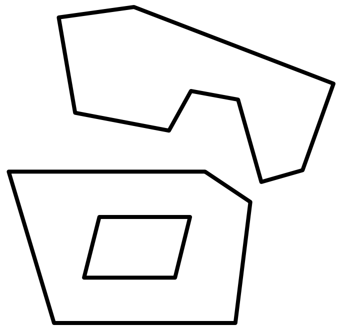
1. What does one do with (convex) 2D polygons with k > 3 sides?



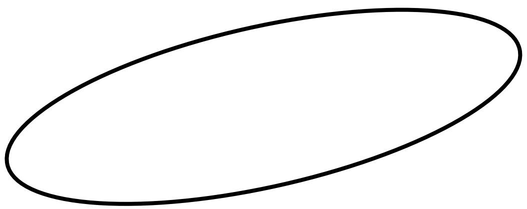
2. Similarly, what does one do with polyhedra in n dimensions with k > n+1 sides? (i.e., non-simplices)



3. What does one do with non-convex (concave) areas?



4. What does one do when the area is not limited by a polyline (piecewise linear curve), but rather by a smooth, closed convex curve?



Generalized Barycentric Coords. with *k*>3 Sides in 2D

- Definition:

  Let Ω be a convex polygon in $\mathbb{R}^2$ with n vertices

  $P_1, \ldots, P_n,\ \ n \geq 3$, in CCW arrangement ("*counter-clockwise*").

  A set of functions $\lambda_i : \Omega \rightarrow \mathbb{R}$ are called barycentric coordinates

  if for all $X \in \Omega$ the following conditions are met:

  1. Partition of unity: $\sum_{i=1}^{n} \lambda_i(X) = 1$
  2. Linear precision: $\sum_{i=1}^{n} \lambda_i(X) P_i = X$
  3. Convex combination: $\forall i = 1 \ldots n:\ \lambda_i(X) \geq 0$

- Other desirable characteristics:
  - "Good behavior" with respect to the area outside of Ω
  - Smoothness: λ_i should be in C^∞
  - Affine invariance

The Interpolation Property

- Theorem:

  Any such (generalized) barycentric coordinates have the interpolation property.

  That is: let data values f_i be given in the vertices P_i ; then the function

  $$f(X) = \sum_{i=1}^{n} \lambda_i(X) \cdot f_i$$

  is actually an interpolating (and not simply approximating) function; it follows that

  $$\forall i : \; f(P_i) = f_i$$

Proof

- We will show:

$$\lambda_i(P_j) = \delta_{ij} \quad (\text{Kronecker Delta})$$

1. Because of properties 1 and 2, one can reproduce all linear functions using barycentric coombinations:
   Let $f(Q)$ be just such a linear function.
   Then

$$f(Q) = \begin{pmatrix} a \\ b \end{pmatrix} Q + c$$

Prop. 2
(lin. prec.)

Prop. 1
(partition of unity)

$$= \begin{pmatrix} a \\ b \end{pmatrix} \sum \lambda_i(Q) P_i + c \sum \lambda_i(Q)$$

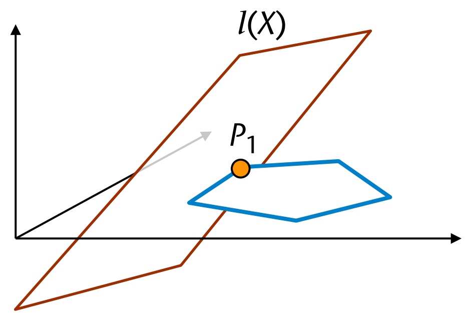
$$= \sum \lambda_i(Q) \left(\begin{pmatrix} a \\ b \end{pmatrix} P_i + c \right)$$

$$= \sum \lambda_i(Q) f(P_i)$$

2. We define plane $l(X)$ so that

$$l(P_1) = 0$$
$$\forall i \geq 2 : \; l(P_i) > 0$$

This is possible because Ω is a convex polygon



3. From step 1, it follows that:

$$l(P) = \sum_{i=1}^{n} \lambda_i(P) \cdot l(P_i) = \sum_{i=2}^{n} \lambda_i(P) \cdot l(P_i)$$

$l(P_1) = 0$

l was specially chosen so that

$l(P_i) > 0$

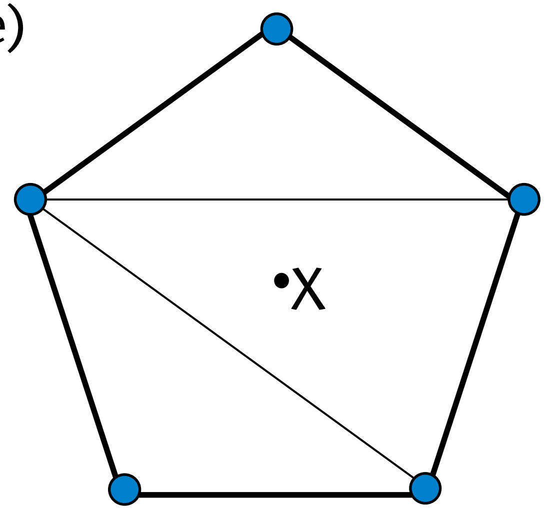
$$0 = l(P_1) = \sum_{i=2}^{n} \lambda_i(P_1) l(P_i) \;\Rightarrow\; \forall i \geq 2 : \; \lambda_i(P_1) = 0$$

4. Because of property 1: $$1 = \sum_{i=1}^{n} \lambda_i(P_1) = \lambda_1(P_1)$$

(step 3)

Trivial Solutions

- Triangulate the polygon (in any way possible)
- Determine the barycentric coordinates with reference to the triangle in which X is located



- Problems:
  - The triangulation is not unique
    - ➢These generalized barycentric coordinates are not unique!
  - These barycentric coordinates are only $\mathcal{C}^0$-continuous
  - The extension for points outside of the polygon is unclear

Construction of Generalized Barycentric Coords.

- Goal: construct barycentric coordinates for any convex polygon in 2D
- Observation: since $\sum \lambda_i = 1$, we have

$$\sum \lambda_i P_i = X \Leftrightarrow \sum \lambda_i \cdot (P_i - X) = 0$$

- If one has functions $w_i = w_i(X)$ for which the following two properties hold,

$$\sum w_i \cdot (P_i - X) = 0 \qquad (1)$$

$$\text{und} \quad \forall i : \; w_i \geq 0 \qquad (2)$$

then one can easily make "real" barycentric coordinates by just letting

$$\lambda_i = \frac{w_i}{\sum_{i=1}^{n} w_i}$$

➢ Look for functions w_i that fulfil conditions (1) & (2)

Notation

- Surface areas:
  - $r_i := \|P_i - X\|$
  - $A_i := A_i(X) = \frac{1}{2}\sin\alpha_i \cdot r_i r_{i+1}$
    $= \mathcal{F}(\Delta X P_i P_{i+1})$
  - $B_i := B_i(X) = -\frac{1}{2} r_{i-1} r_{i+1} \sin(\alpha_{i-1} + \alpha_i)$
    $= \mathcal{F}(\Delta X P_{i+1} P_{i-1})$
    - Note: the sign of B_i is *negative,* if X is outside of $\Delta P_{i-1} P_i P_{i+1}$
  - $C_i := C_i(X) = \mathcal{F}(\Delta P_{i-1} P_i P_{i+1})$
    $= A_i + A_{i-1} + B_i$
- Indices: in the following, all indices are meant to be "modulo n", i.e. $P_i := P_{i \bmod n}$ and so $P_{n+1} = P_1$, $P_{-1} = P_{n-1}$